# EULER-MACLAURIN FORMULA FOR THE MULTIPLICITIES OF THE INDEX OF TRANSVERSALLY ELLIPTIC OPERATORS

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ABSTRACT. Let $G$ be a connected compact Lie group acting on a manifold $M$ and let $D$ be a transversally elliptic operator on $M$. The multiplicity of the index of $D$ is a function on the set $\hat{G}$ of irreducible representations of $G$. Let $T$ be a maximal torus of $G$ with Lie algebra $\mathfrak{t}$. We construct a finite number of piecewise polynomial functions on $\mathfrak{t}^*$, and give a formula for the multiplicity in term of these functions.

## 1. Introduction

Let $G$ be a compact connected Lie group acting on a manifold $M$. Atiyah-Singer [1] have associated to any $G$-transversally elliptic pseudo-differential operator $D$ on $M$ a virtual trace class representation of $G$ :

$$\mathrm{Index}(D) = \sum_{\lambda \in \hat{G}} \mathrm{mult}(D)(\lambda) V_\lambda.$$

If $D$ is elliptic, the preceding sum of irreducible representations $V_\lambda$ of $G$ is finite. As shown in [1], the computation of the function $\mathrm{mult}(D) : \hat{G} \to \mathbb{Z}$ can be reduced to the case where $G$ is a torus.

Thus we assume that $G$ is a torus with Lie algebra $\mathfrak{g}$, and parameterize the irreducible representations of $G$ by the lattice $\Lambda \subset \mathfrak{g}^*$ of characters of $G$. Assume, in this introduction, that the stabilizer of any point of $M$ is connected. We construct a particular "spline" function $m$ on $\mathfrak{g}^*$, extending the function $\mathrm{mult}(D)$ in a piecewise polynomial function on $\mathfrak{g}^*$. The construction of the function $m$ is based on the notion of infinitesimal index [8] and is canonically associated to the equivariant Chern character $\mathrm{ch}(\sigma)$ of the principal symbol $\sigma$ of $D$.

A trivial example is when $G$ acts on $G = M$ by left translations and $D$ is the operator 0, with index

$$L^2(G) = \sum_{\lambda \in \Lambda} e^{i\lambda}.$$

Then the function $\mathrm{mult}(D)$ is identically equal to 1 on $\Lambda$, and extended by the constant function $m = 1$.

The construction of $m$ was the object of the article [7], in the case where $M$ is a vector space with a linear action of $G$. In this note, we just state the results. Our proofs are very similar to those of [7]. We use results of [1] (see

also [9]) on generators of the equivariant $K$-theory. Then our main tool is (as in [7]) the Dahmen-Micchelli inversion formula for convolution with the box spline, a "Riemann-Roch formula" for approximation theory ([5], see [7]).

Let us briefly explain the origin of our construction. Denote by $\mathrm{Index}(D)(g)$ the corresponding generalized function.

$$\mathrm{Index}(D)(g) = \sum_{\lambda\in\hat{G}} \mathrm{mult}(D)(\lambda)\mathrm{Tr}_{V_\lambda}(g)$$

on $G$. Recall the formula

$$\mathrm{Index}(D)(\exp X) = \frac{1}{(2i\pi)^{\dim M}} \int_{T^*M} \frac{\mathrm{ch}(\sigma)(X)}{J(M)(X)}, \tag{1}$$

obtained in ([4], [3]), a "delocalized" version of Atiyah-Bott-Segal-Singer equivariant index formula. Here the inverse of $J(M)$ is the equivariant Todd class of $T^*M$ (considered as an almost complex manifold). The formula above is valid when $X$ varies in a neighborhood of 0 in the Lie algebra $\mathfrak{g}$ of $G$ as an equality of generalized functions of $X$.

Here we replace the equivariant class $J(M)$ by its formal series of homogeneous components. By Fourier transform, we obtain a series of generalized functions on $\mathfrak{g}^*$. In a limit sense explained in this note, the restriction of this series to $\Lambda$ coincides with the function $\lambda \to \mathrm{mult}(D)(\lambda)$.

In a subsequent note, we will give a geometric interpretation of the piecewise polynomial function $m$, when $D$ is a Dirac operator twisted by a line bundle $L$ in term of the moment map associated to an equivariant connection on $L$.

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## 2. Transversally elliptic symbols

Let $G$ be a compact Lie group. Let $M$ be a $G$-manifold, and $T^*M$ its cotangent bundle. We denote by $p : T^*M \to M$ the projection.

Let $\omega$ be the Liouville one form: for $x \in M$, $\xi \in T^*_xM$ and $V$ a tangent vector at the point $(x, \xi) \in T^*M$, $\omega_{x,\xi}(V) = \langle \xi, p_*V\rangle$. By definition, the symplectic form $\Omega = -d\omega$ is the symplectic form of $T^*M$, and we use the corresponding orientation of $T^*M$ to compute integrals on $T^*M$ of differential forms with compact support.

Let $\mathcal{E}^\pm$ be two $G$-equivariant complex vector bundles over $M$. A $G$-equivariant bundle map $\sigma : p^*\mathcal{E}^+ \to p^*\mathcal{E}^-$ will be called a symbol. The support $\mathrm{supp}(\sigma) \subset T^*M$ of $\sigma$ is the set of elements $n \in T^*M$ such that the linear map $\sigma(n) : \mathcal{E}^+_{p(n)} \to \mathcal{E}^-_{p(n)}$ is not invertible. An elliptic symbol is a symbol such that $\mathrm{supp}(\sigma)$ is compact.

Denote by $T^*_GM \subset T^*M$ the union of the space of covectors conormal to the $G$ orbit through $x$. If $X \in \mathfrak{g}$, we denote by $v_X$ the vector field on $N$ generated by $-X \in \mathfrak{g}$. Let $\mu : T^*M \to \mathfrak{g}^*$ be the moment map

$\langle\mu(x,\xi),X\rangle = -\langle\xi,v_X\rangle$. Then the zero fiber $Z=\mu^{-1}(0)$ of the moment map $\mu$ is the space $T^*_G M$. If the support of the symbol $\sigma$ intersects $Z$ in a compact set, we say that $\sigma$ is a transversally elliptic symbol (it is elliptic in the directions transverse to the $G$-orbits). Then $\sigma$ determines an element in the topological equivariant $K$ group $K^0_G(Z)$, still denoted by $\sigma$.

If $D$ is a pseudo-differential operator with principal symbol $\sigma$, we say that $D$ is transversally elliptic if its principal symbol is transversally elliptic. The index of $D$ depends only of the class of $\sigma \in K^0_G(Z)$.

Atiyah-Singer [1] have associated to any transversally elliptic symbol $\sigma$ a virtual trace class representation of $G$ :

$$\mathrm{Index}(\sigma) = \sum_{\lambda\in\hat{G}} \mathrm{mult}(\sigma)(\lambda) V_\lambda.$$

If $\sigma$ is elliptic, the preceding sum of irreducible representations $V_\lambda$ of $G$ is finite. By construction, if $D$ is a pseudo-differential operator with principal symbol $\sigma$, then $\mathrm{Index}(D) = \mathrm{Index}(\sigma)$ (and $\mathrm{mult}(D) = \mathrm{mult}(\sigma)$).

Let us give examples of such operators.

**Example 2.1.** Elliptic operators are transversally elliptic. If $M$ is an even dimensional compact manifold and is oriented, then, up to stable homotopy, any elliptic symbol is the principal symbol of a twisted Dirac operator.

**Example 2.2.** Let $M = G$ and let $H \subset G$ be a compact connected subgroup of $G$. Then the 0 operator is transversally elliptic with respect to the action of $G\times H$, acting by left and right multiplication, and $\mathrm{Index}(0) = \sum_{\lambda\in\hat{G},\mu\in\hat{H}} m(\lambda,\mu) V^G_\lambda \otimes V^H_\mu$, where $m(\lambda,\mu)$ is the multiplicity of the irreducible representation $V^H_\mu$ of $H$ in the irreducible representation $(V^G_\lambda)^*$ of $G$.

**Example 2.3.** Let $M$ be a compact manifold and let $G$ be a group acting on $M$, and such that $G$ acts infinitesimally freely on $M$. Then $M/G$ is an orbifold. Let $\sigma$ be a $G$-transversally elliptic symbol on $M$. Denote by $\lambda_0$ the trivial representation of $G$. If $D$ is a $G$-invariant operator with principal symbol $\sigma$, $\mathrm{mult}(\sigma)(\lambda_0)$ is the index of the elliptic operator associated to $D$ on the orbifold $M/G$.

**Example 2.4.** Let $V$ be a Hermitian vector space and $G$ a subgroup of $U(V)$. Let $c$ be the Clifford representation of $V^*$ in the superspace $E = \Lambda V^*$.

Consider the moment map $\Phi : V \to \mathfrak{g}^*$. Assume $\Phi^{-1}(0) = \{0\}$. Identify the Kirwan vector field associated to the moment map $\Phi$ to a $G$-invariant one form $\lambda$ on $V$. Then the pushed symbol $\sigma(v,\xi) = c(\xi+\lambda_v)$ is a transversally elliptic symbol on $V$. The function $\mathrm{mult}(\sigma)(\lambda)$ is the multiplicity of the representation $V_\lambda$ of $G$ in the space $S(V^*)$ of polynomial functions on $V$ (see [6]). In particular, if $G$ is a torus, then $\lambda \to \mathrm{mult}(\sigma)(\lambda)$ is the vector partition function associated to the list of weights of $G$ in $V$.

## 3. Definitions

3.1. **Equivariant cohomology.** Let $G$ be a compact Lie group, with Lie algebra $\mathfrak{g}$, acting on a manifold $N$. Let $\mathcal{A}(N)$ be the space of differential forms on $N$. If $X \in \mathfrak{g}$, we denote by $\iota_X$ the contraction of a differential form on $N$ by $v_X$. Let $D = d - \iota(v_X)$ be the equivariant differential. In the Cartan model, a representative of an element of the equivariant cohomology $H^*_G(N)$ is an equivariant form $\alpha : \mathfrak{g} \to \mathcal{A}(N)$ such that $D(\alpha) = 0$. The dependance of $\alpha$ in $X \in \mathfrak{g}$ is polynomial. Then $H^*_G(N)$ is a $\mathbb{Z}$-graded algebra.

Let $q$ be a formal variable. If $F$ is a vector space and $f([q]) = \sum_{k=0}^{\infty} q^k f_k$ is a formal series of elements of $F$, we write $f \in F[[q]]$. If $f(q,x)$ is a smooth function of $q$, defined near $q = 0$, and depending of some parameters $x$, we denote by $f([q],x) = \sum_{k=0}^{\infty} q^k f_k(x)$ its Taylor series at $q = 0$, a formal series of functions of $x$. If the series $f([q])$ is finite, we write $f([1])$ of $f([q])_{q=1}$ for the sum $\sum_k f_k$.

We introduce formal series

$$\alpha([q], X) = \sum_{k=0}^{\infty} q^k \alpha_k(X)$$

of equivariant forms on $N$. If the constant term $\beta_0$ of $\beta([q], X) = \sum_{k=0}^{\infty} q^k \beta_k(X)$ is a non zero constant, we can define the formal series $\alpha([q], X)/\beta([q], X)$. If $N$ is oriented, we can integrate such series against any equivariant form $c(X)$ with compact support, and obtain a formal power series $\int_N c(X)\alpha([q], X) = \sum_{k=0}^{\infty} q^k \int_N c(X)\alpha_k(X)$ of functions of $X \in \mathfrak{g}$.

If $Z$ is a $G$-invariant closed subset of $N$, we have defined in [8] a Cartan model for the space $H^{\infty}_{G,c}(Z)$ of equivariant cohomology with compact supports. A representative is an equivariant form $\alpha : \mathfrak{g} \to \mathcal{A}_c(N)$ such that $D(\alpha) = 0$ in a neighborhood of $Z$. Here $\mathcal{A}_c(N)$ is the space of differential forms with compact supports and the dependance of $\alpha$ in $X$ is $C^{\infty}$.

Let $M$ be a $G$-manifold, $T^*M$ its cotangent bundle and $Z = T^*_G M$. If $\alpha$ is an equivariant cohomology class on $M$, its pull back $p^*\alpha$ is an equivariant cohomology class on $T^*M$, that we denote simply by $\alpha$.

Let $\sigma : p^*\mathcal{E}^+ \to p^*\mathcal{E}^-$ be a transversally elliptic symbol. Choosing $G$-invariant connections on $p^*\mathcal{E}^{\pm}$, coinciding via $\sigma$ outside a small neighborhood of $\mathrm{supp}(\sigma)$, we can construct the equivariant Chern character $\mathrm{ch}(\sigma) \in H^{\infty}_{G,c}(Z)$ of $\sigma$, an equivariant cohomology class, represented by a differential form with compact support on $T^*M$, still denoted by $\mathrm{ch}(\sigma)$.

Let $\mathcal{V} \to M$ be a real or complex vector bundle on $M$ with typical fiber a vector space $V$. The Chern-Weil map $W$ associates to an invariant polynomial $f$ on $\mathrm{End}(V)$ an equivariant characteristic class $W(f)$ in $H^*_G(M)$. If $f$ is homogeneous of degree $k$, then $W(f)$ is homogeneous of degree $2k$. Our conventions for the Chern-Weil homorphism $W$ and the Chern character are as in [2].

Let $A \in \mathrm{End}(V)$. Introduce a variable $q$, and consider the Taylor expansion

$$\det_V\left(\frac{e^{qA}-1}{qA}\right)=\sum_{k=0}^{\infty}q^kT_k(A)=1+\frac{q}{2}\mathrm{Tr}(A)+\cdots.$$

Thus $T_k$ is an invariant homogeneous polynomial of degree $k$ on $\mathrm{End}(V)$.

Our main new concept is the introduction of the following formal equivariant characteristic class of $M$.

**Definition 3.2.** The formal $J$-class of $M$ is the series of elements of $H^*_G(M)$ defined by

$$J([q],M)=\sum_{k=0}^{\infty}q^kW(T_k)$$

obtained by applying the Chern-Weil map for the real vector bundle $\mathcal{V}=TM\to M$ to the series $\det_V(\frac{e^{qA}-1}{qA})=\sum_{k=0}^{\infty}q^kT_k(A)$.

Here $W(T_k)$ is homogeneous of degree $2k$.

When $G=\{1\}$, then $p^*J([1],M)=J(M)$ is the inverse of the usual Todd class of the tangent bundle to $T^*M$ (considered as an almost complex manifold). Furthermore, if $\sigma$ is elliptic, Atiyah-Singer formula for $\mathrm{Index}(\sigma)\in\mathbb{Z}$ is

$$\mathrm{Index}(\sigma)=\frac{1}{(2i\pi)^{\dim M}}\int_{T^*M}\frac{\mathrm{ch}(\sigma)}{J(M)}.$$

3.3. **Piecewise polynomial generalized functions.** In this subsection, $G$ is a torus.

Let $V=\mathfrak{g}^*$ equipped with the lattice $\Lambda\subset\mathfrak{g}^*$ of weights of $G$. If $g=\exp X$, we denote by $g^\lambda=e^{i\langle\lambda,X\rangle}$. The function $g\mapsto g^\lambda$ is a character of $G$.

We denote by $\mathcal{C}(\Lambda)$ the space of (complex valued) functions on $\Lambda$. If $g\in G$, we denote by $\hat{g}$ the function $g\to g^\lambda$ on $\Lambda$. If $m\in\mathcal{C}(\Lambda)$, then $\hat{g}m\in\mathcal{C}(\Lambda)$ is defined by $(\hat{g}m)(\lambda)=g^\lambda m(\lambda)$.

Using the Lebesgue measure $d\xi$ associated to $\Lambda$, we identify generalized functions on $\mathfrak{g}^*$ and distributions on $\mathfrak{g}^*$. Let $f$ be a test function on $\mathfrak{g}^*$, we define $\hat{f}(X)=\int_{\mathfrak{g}^*}e^{i\langle\xi,X\rangle}f(\xi)d\xi$. The Lebesgue measure $dX$ on $\mathfrak{g}$ is such that the Fourier inversion $\int_{\mathfrak{g}}e^{-i\langle\xi,X\rangle}\hat{f}(X)dX=f(\xi)$ holds. If $h$ is a generalized function on $\mathfrak{g}^*$, we denote by $\int_{\mathfrak{g}^*}h(\xi)f(\xi)d\xi$ its value on the test function $f$. We denote by $\delta_v$ the Dirac function at the point $v\in V$.

Let $\mathcal{H}$ be a finite collection of rational hyperplanes in $\mathfrak{g}^*$. An element of $\mathcal{H}$ will be called an admissible hyperplane. An element $v\in V$ is called $\mathcal{H}$-*generic* if $v$ is not on any hyperplane of the collection $\mathcal{H}$. We just say that $v$ is generic.

An admissible wall is a translate of an hyperplane in $\mathcal{H}$ by an element of $\Lambda$. A *tope* $\tau$ is a connected component of the complement of all admissible walls and we denote by $V_{reg}$ the union of topes.

A piecewise polynomial function is a function on $V_{reg}$ which is given by a polynomial formula on each tope. We denote by $PW$ the space of piecewise polynomial functions.

Consider $f \in PW$ (defined on $V_{reg}$) as a locally $L^1$-function on $V$, thus $f$ defines a generalized function on $V$. An element of $PW$, considered as a generalized function on $V$, will be called a piecewise polynomial generalized function.

**Definition 3.4.** The space $\mathcal{S}$ is the space of generalized functions on $V$ generated by the action of constant coefficients differential operators on piecewise polynomial generalized functions.

For example, the Heaviside function on $\mathbb{R}$ is a piecewise polynomial function. Its derivative in the sense of generalized functions is the Dirac function at 0.

A function in $\mathcal{S}$ can be evaluated at any point of $V_{reg}$. If $v \in V$, and $\epsilon$ is a generic vector, then $v + t\epsilon$ is in $V_{reg}$ if $t > 0$ and sufficiently small.

**Definition 3.5.** Let $v \in V$, and $f \in \mathcal{S}$. Let $\epsilon$ a generic vector. Define $(\lim_\epsilon f)(v) = \lim_{t>0,t\to 0} f(v + t\epsilon)$.

Remark that this definition depends only of the restriction of $f$ to $V_{reg}$.

Introduce formal series $m([q]) = \sum_{k=0}^{\infty} q^k m_k$ of generalized functions on $V$. Then if $f$ is a test function

$$\int_{\mathfrak{g}^*} m([q])(\xi) f(\xi) d\xi = \sum_{k=0}^{\infty} q^k \int_{\mathfrak{g}^*} m_k(\xi) f(\xi) d\xi$$

is a formal power series in $q$. It may be evaluated at $q = 1$ if the preceding series is finite (or convergent).

If $\epsilon$ is generic, we define a map $\lim_\epsilon^\Lambda : \mathcal{S}[[q]] \to \mathcal{C}(\Lambda)[[q]]$ by

$$(\lim_\epsilon^\Lambda m([q]))(\lambda) = \sum_{k=0}^{\infty} q^k (\lim_\epsilon m_k)(\lambda).$$

If all, but a finite number, the functions $m_k$ are equal to 0 on $V_{reg}$, then $\lim_\epsilon^\Lambda m([q])|_{q=1}$ is an element of $\mathcal{C}(\Lambda)$.

Formal series of distributions occur naturally in the context of Euler-MacLaurin formula.

3.6. **Fourier transforms of equivariant integrals.** In this section, $G$ is a torus acting on a manifold $M$, $V = \mathfrak{g}^*$ and $Z = T_G^* M$. We assume that $M$ admits a $G$-equivariant embedding in a vector space provided with a linear representation of $G$. For $x \in M$, denote by $\mathfrak{g}_x \subset \mathfrak{g}$ the infinitesimal stabilizer of $x \in M$. We assume that the generic infinitesimal stabilizer for the action of $G$ on $M$ is equal to 0. We denote by $\mathcal{I}^1$ the set of infinitesimal stabilizers of dimension 1. Let $\mathcal{H}$ be the finite collection of hyperplanes $\ell^\perp$ where the line $\ell$ varies in $\mathcal{I}^1$. Let $\mathcal{S}$ be the corresponding space of generalized functions on $V = \mathfrak{g}^*$.

Consider a transversally elliptic symbol $\sigma$. We then use the notion of infinitesimal index to perform the integration on $T^*M$.

If $\alpha$ is a closed equivariant form with **polynomial** coefficients, and $f$ a test function on $\mathfrak{g}^*$, the double integral

$$\int_{T^*M}\int_{\mathfrak{g}} e^{isD\omega(X)}\mathrm{ch}(\sigma)(X)\alpha(X)\hat{f}(X)dX$$

is independent of $s \in \mathbb{R}$, for $s$ positive and sufficiently large. We denote it by

$$\int_{T^*M}^{\omega}\int_{\mathfrak{g}} \mathrm{ch}(\sigma)(X)\alpha(X)\hat{f}(X)dX.$$

**Definition 3.7.** If $\sigma$ is a transversally elliptic symbol and $\alpha$ is a closed equivariant form with **polynomial** coefficients, we define the generalized function $m(\sigma,\alpha)$ on $\mathfrak{g}^*$ so that

$$\int_{T^*M}^{\omega}\int_{\mathfrak{g}} \mathrm{ch}(\sigma)(X)\alpha(X)\hat{f}(X)dX = \int_{\mathfrak{g}^*} m(\sigma,\alpha)(\xi)f(\xi)d\xi$$

for any test function $f(\xi)$ on $\mathfrak{g}^*$.

Then $m(\sigma,\alpha)$ depends only of the cohomology class of $\alpha$ (still denoted by $\alpha$).

**Proposition 3.8.** • *The generalized function $m(\sigma,\alpha)$ belongs to $\mathcal{S}$.*
• *If $\alpha$ is an homogeneous equivariant class of degree $2k$ , then $m(\sigma,\alpha)$ restricts to a polynomial of degree less or equal to $\dim M - \dim G - k$ on each connected component of $V_{reg}$. In particular, when $k$ is greater that $\dim M - \dim G$, $m(\sigma,\alpha)$ restricts to $0$ on $V_{reg}$.*

## 4. The elliptic case

Let $G$ be a torus acting on a connected manifold $M$. To explain the flavor of our formula, we assume that $\sigma$ is an elliptic symbol and we make a further simplification. We assume that the stabilizer of any point of $M$ is connected and that the generic stabilizer is trivial.

We write

$$\mathrm{Index}(\sigma) = \sum_{\lambda\in\Lambda} \mathrm{mult}(\sigma)(\lambda)e^{i\lambda}$$

with $\mathrm{mult}(\sigma) \in \mathcal{C}(\Lambda)$.

To the elliptic symbol $\sigma$, we associate a series $m([q],\sigma)$ of generalized functions on $\mathfrak{g}^*$.

**Definition 4.1.** Define $m([q],\sigma) = \sum_{k=0}^{\infty} q^k m_k$ to be the series of generalized functions on $\mathfrak{g}^*$ such that, for any test function $f$ on $\mathfrak{g}^*$,

$$(2)\quad (2i\pi)^{-\dim M}\int_{T^*M}\int_{\mathfrak{g}} \frac{\mathrm{ch}(\sigma)(X)}{J([q],M)(X)}\hat{f}(X)dX = \int_{\mathfrak{g}^*} m([q],\sigma)(\xi)f(\xi)d\xi.$$

All distributions $m_k$ are compactly supported, as we are in the elliptic case. The restriction of $m_k$ to each connected component of $V_{reg}$ is a polynomial of degree less or equal to $\dim M - \dim G - k$.

Compare with Formula (1) (for $D$ with principal symbol $\sigma$). The equivariant form $\frac{\mathrm{ch}(\sigma)(X)}{J([q],M)(X)}$ is for $q=1$ equal to $\frac{\mathrm{ch}(\sigma)(X)}{J(M)(X)}$. The left hand side of the equality (2) cannot be evaluated for $q=1$, as $J(M)(X)$ is not invertible for $X$ large. Here comes the miracle. The right hand side can be evaluated at $q=1$, when restricted to $\Lambda$ and we have the following theorem.

**Theorem 4.2.** *For any generic vector $\epsilon$,*

$$\mathrm{mult}(\sigma) = \lim_{\epsilon}^{\Lambda} m([q],\sigma)|_{q=1}.$$

When $f$ is a polynomial, $\hat{f}(X)$ is supported at 0, and the two formulae (1), (2) coincide at $q=1$, thus we have the following Euler-MacLaurin formula.

**Theorem 4.3.** *For any polynomial function $f$ on $\mathfrak{g}^*$, $\int_{\mathfrak{g}^*} m_k(\xi)f(\xi)d\xi$ is equal to 0 when $k$ is sufficiently large, and*

$$\sum_{\lambda} \mathrm{mult}(\sigma)(\lambda)f(\lambda) = \sum_{k=0}^{\infty}\int_{\mathfrak{g}^*} m_k(\xi)f(\xi)d\xi.$$

Let us give a simple example. Let $M = P_1(\mathbb{C})$, let $A$ be a non negative integer and $\overline{\partial}_A$ the $\overline{\partial}$ operator on sections of $\mathcal{L}^A$, where $\mathcal{L}$ is the dual of the tautological bundle. Let $S^1$ be the circle group with generator $J$, acting on $[x,y] \in P^1(\mathbb{C})$ by $\exp(\theta J)[x,y] = [e^{i\theta}x, y]$. Here $[x,y]$ are the homogeneous coordinates on $P_1(\mathbb{C})$. Then

$$\mathrm{Index}(\overline{\partial}_A)(\exp(\theta J)) = \sum_{d=0}^{A} e^{id\theta}.$$

We identify $\mathfrak{g}^*$ with $\mathbb{R}$ and $\Lambda$ with $\mathbb{Z}$. The multiplicity function $\mathrm{mult}(\overline{\partial}_A)$ is such that $\mathrm{mult}(\overline{\partial}_A)(n) = 1$ of $0 \leq n \leq A$, otherwise is equal to 0.

Consider the function

$$F(q,\theta) = \frac{(1-e^{-i\theta})}{(1-e^{iq\theta})(1-e^{-iq\theta})} + \frac{e^{iA\theta}(1-e^{i\theta})}{(1-e^{iq\theta})(1-e^{-iq\theta})}.$$

The Taylor series $F([q],\theta)$ is a series of analytic functions of $\theta$ and it follows from the localisation formula on $T^*P_1(\mathbb{C})$ that, for $X = \theta J$,

$$(2i\pi)^{-\dim M}\int_{T^*M}\frac{\mathrm{ch}(\sigma)(X)}{J([q],M)(X)} = F([q],\theta).$$

Let us write the first terms of the expansion of $F([q],\theta)$. This is

$$(e^{iA\theta} + 1 - e^{i(A+1)\theta} - e^{-i\theta})(-\frac{1}{i^2\theta^2} + \frac{1}{12}q^2 - \frac{1}{240}\, i^2\theta^2 q^4 + \frac{1}{6048}\, i^4\theta^4 q^6 + \cdots)$$

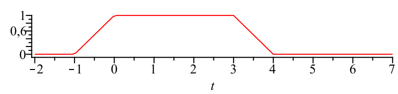


FIGURE 1. $m_0(\xi)$ for $A = 3$

The Fourier transform of $F([q], \theta)$ is the series of generalized functions

$$m([q], \xi) = \sum_{k=0}^{\infty} q^k m_k(\xi).$$

We see that only $m_0(\xi)$ is non zero on $\mathbb{R} \setminus \mathbb{Z}$, and given by the piecewise polynomial function

$$m_0(\xi) = \begin{cases} 0 & \xi \leq -1 \\ \xi + 1 & -1 \leq \xi \leq 0 \\ 1 & 0 \leq \xi \leq A \\ (A+1) - \xi & A \leq \xi \leq A+1 \\ 0 & (A+1) \leq \xi \end{cases}$$

Remark that $m_0(\xi)$ continuous and coincide with the function $\mathrm{mult}(\overline{\partial}_A)$ on $\mathbb{Z}$. This is the content of Theorem 4.2.

We can compute further terms of the expansion,

$$m([q], \xi) = m_0(\xi) + \frac{q^2}{12}(\delta_0 + \delta_A - \delta_{-1} - \delta_{A+1}) - \frac{q^4}{240}(\frac{d}{d\xi})^2(\delta_0 + \delta_A - \delta_{-1} - \delta_{A+1}) + \cdots$$

Then for any polynomial function $f$, we obtain a version of Euler-MacLaurin formula:

$$\sum_{d=0}^{A} f(d) = \int_{\mathbb{R}} f(\xi) m_0(\xi) d\xi$$

$$+\frac{1}{12}(f(0)+f(A)-f(-1)-f(A+1))-\frac{1}{240}(f''(0)+f''(A)-f''(-1)-f''(A+1))+\cdots$$

This is the content of Theorem 4.3.

## 5. The general formula for a torus

Let $G$ be a torus acting on $M$. The collection of hyperplanes $\mathcal{H}$ in $\mathfrak{g}^*$ is as in Subsection 3.6 and $Z = T^*_G M$.

Let $\sigma \in p^*\mathcal{E}^+ \to p^*\mathcal{E}^-$ be a transversally elliptic symbol on $M$. We assume that the generic infinitesimal stabilizer for the action of $G$ on $M$ is equal to 0. We can always reduce the problem to this case.

For $g \in G$, denote by $M^g = \{x \in M; gx = x\}$. This is a manifold, which might not be connected. We say that $g$ is a vertex if there exists $x \in M^g$ with a finite stabilizer under the action of $G$ (if stabilizers are all connected, then the only vertex is $g = 1$). We denote by $\mathcal{V}(M)$ the set of vertices of the action of $G$ on $M$. This is a finite subset of $G$.

Let $g \in G$. Then $g$ acts by a fiberwise transformation on $\mathcal{E}^\pm \to M^g$ still denoted $g$. The morphism $\sigma$ commutes with $g$ over $T^*M^g$. The equivariant twisted Chern character $\mathrm{ch}^g(\sigma)$ is defined in [7] as an element of $H^\infty_{G,c}(Z^g)$.

If $\alpha \in H^*_G(M^g)$, we can define a generalized function $m(g,\sigma,\alpha)$ on $\mathfrak{g}^*$ by the formula

$$\int^{\omega}_{T^*M^g}\int_{\mathfrak{g}} \mathrm{ch}^g(\sigma)(X)\alpha(X)\hat{f}(X)dX = \int_{\mathfrak{g}^*} m(g,\sigma,\alpha)(\xi)f(\xi)d\xi$$

for any test function $f$ on $\mathfrak{g}^*$.

**Lemma 5.1.** *The generalized function $m(g,\sigma,\alpha)$ belong to $\mathcal{S}$.*

*If $g$ is not a vertex, the generalized function $m(g,\sigma,\alpha)$ vanishes on $V_{reg}$.*

If $E$ is a vector space and $s \in End(E)$ is an invertible and semi-simple transformation of $E$, we denote by $GL(s)$ the group of invertible linear transformation of $E$ commuting with $s$. We consider for $A \in End(E)$

$$D(q,s,A) = \det_E(1 - se^{qA}),$$

an analytic function of $A \in End(E)$. Write the Taylor series

$$D([q],s,A) = \sum_{k=0}^{\infty} q^k D^s_k(A).$$

Then $A \to D^s_k(A)$ are homogeneous polynomials of degree $k$, invariant under $GL(s)$.

Consider the normal bundle $\mathcal{N} \to M^g$. Thus $g$ produces an invertible linear transformation of $\mathcal{N}_x$ at any $x \in M^g$. The Chern Weil homomorphism for the bundle $\mathcal{N}$ (with structure group $GL(g)$) produces a series $D([q],g,M/M^g) := \sum_{k=0}^{\infty} q^k W(D^g_k)$ of closed equivariant forms on $M^g$. The coefficient in $q^0$ of this series is just the function $x \to \det_{\mathcal{N}_x}(1-g)$, a function which is a non zero constant on each connected component of $M^g$. Similarly the function $\dim M^g$ is constant on each connected component of $M^g$.

Recall that we have defined the class of $J([q], M^g)$ on $M^g$.

**Definition 5.2.** Define the series of generalized functions $m([q], g, \sigma)$ of generalized functions on $\mathfrak{g}^*$ such that

$$\int_{T^*M^g}^{\omega} \int_{\mathfrak{g}} (2i\pi)^{-\dim M^g} \frac{\mathrm{ch}^g(\sigma)(X)}{J([q], M^g)(X) D([q], g, M/M^g)(X)} \hat{f}(X) dX$$

$$= \int_{\mathfrak{g}^*} m([q], g, \sigma)(\xi) f(\xi) d\xi$$

for any test function $f(\xi)$ on $\mathfrak{g}^*$.

Here is the multiplicity formula for the index of a transversally elliptic symbol $\sigma$.

**Theorem 5.3.** *For any $\epsilon$ generic*

$$\mathrm{mult}(\sigma) = \sum_{g \in \mathcal{V}(M)} \hat{g} \lim_{\epsilon}^{\Lambda} m([q], g^{-1}, \sigma)|_{q=1}.$$

## 6. Compact groups

Let $G$ be a compact simply connected Lie group acting on a manifold $M$ and let $Z = T_G^* M$. Let $T$ be a maximal torus of $G$. The action of $T$ on $M$ defines a set of hyperplanes $\mathcal{H}$ in $\mathfrak{t}^*$ and a space $\mathcal{S}$ of distributions on $\mathfrak{t}^*$. We denote by $\mathcal{V}(M) \subset T$ the set of vertices for the action of $T$ on $M$.

We parameterize the set of irreducible representations of $G$ as follows. We consider $\Lambda \subset \mathfrak{t}^*$ to be the set of weights of $T$. We choose a system of positive roots $\Delta^+ \subset \mathfrak{t}^*$. Let $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$, and $\mathfrak{t}^*_{>0}$ be the positive Weyl chamber. For $\lambda \in \Lambda$, regular and dominant, we denote by $V_\lambda$ the irreducible representation of $G$ of highest weight $\lambda - \rho$.

Let $\sigma \in K_G^0(Z)$. We write

$$\mathrm{Index}(\sigma) = \sum_{\lambda \in \Lambda \cap \mathfrak{t}^*_{>0}} \mathrm{mult}(\sigma)(\lambda) V_\lambda$$

and we extend the function $\mathrm{mult}(\sigma)$ in an antiinvariant (under the action of the Weyl group $W$) function $\tilde{\mathrm{mult}}(\sigma)$ on $\Lambda$. The function $\tilde{\mathrm{mult}}(\sigma)$ is the multiplicity index of a $T$-transversally elliptic symbol $\tilde{\sigma}$ on $M$, constructed in [1]. (If $\sigma$ is itself $T$ transversally elliptic, denote by $\sigma^T$ the symbol $\sigma$ considered as a $T$-transversally elliptic symbol. Then $\tilde{\sigma}$ is the symbol $\sigma^T$ twisted by the representation of $T$ in the spinor superspace of $\mathfrak{g}/\mathfrak{t}$, and then $\tilde{\mathrm{mult}}(\sigma) = \sum_{w \in W} \epsilon(w) \mathrm{mult}(\sigma^T)(\lambda + w\rho)$.)

The multiplicity of $\tilde{\sigma}$ is an anti-invariant function on $\Lambda$. Thus we construct from $\tilde{\sigma}$ the series $m([q], g, \tilde{\sigma})$ of anti-invariant generalized functions on $\mathfrak{t}^*$. belonging to the space $\mathcal{S}$.

**Theorem 6.1.** • *For any $\epsilon$ generic,*

$$\tilde{\mathrm{mult}}(\sigma) = \sum_{g \in \mathcal{V}(M)} \hat{g} \overset{\Lambda}{\lim_{\epsilon}} (m([q], g, \tilde{\sigma}))|_{q=1}.$$

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